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# Generalized Set-Valued Variational Inclusions and Resolvent Equations in Banach Spaces

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**Abstract**—In this paper, we construct a new iterative algorithm for set-valued variational inclusions without the compactness condition and study the convergence of the perturbed Ishikawa iterative process for solving a class of the generalized single-valued variational inclusions in Banach spaces. The result obtained in this paper is a generalization and improvement of Noor's theorem [1].


**Keywords**—Variational inclusions, Resolvent operators, Iterative algorithms, Convergence criteria.

## 1. INTRODUCTION

In recent years, variational inequalities have been extended and generalized in different directions using novel and innovative techniques both for its own sake and for its applications. A useful and an important generalization of variational inequalities is a variational inclusion. In 1998, using the concept and technique of resolvent operators, Noor [1] introduced and studied a new class of variational inclusions in a Hilbert space $H$, which is called the generalized set-valued variational inclusion.

For a given maximal monotone mapping $A : H \to H$, a nonlinear mapping $N(\cdot,\cdot) : H \times H \to H$, set-valued mappings $T, V : H \to C(H)$, and a single-valued map $g : H \to H$, find $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 = N(w,y) + A(g(u)),$$

where $C(H)$ denotes the family of all nonempty compact subsets of $H$.

In this paper, we study a class of more general set-valued variational inclusions without the compactness condition in Banach spaces. And, for solving a class of the generalized single-valued variational inclusions in Banach spaces, we construct a new iterative algorithm, which is called the perturbed Ishikawa iterative process. The results presented in this paper generalize, improve, and unify the corresponding results of Chang [2], Hassouni and Moudafi [3], Huang [4,5], Noor [1,6–9], and Zeng [10].

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## 2. PRELIMINARIES

Let $E$ be a real Banach space, $E^*$ be the topological dual space of $E$, $CB(E)$ be a family of nonempty bounded closed subsets of $E$, $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right\},$$

$\langle\cdot,\cdot\rangle$ be the dual pair between $E$ and $E^*$, $D(T)$ denotes the domain of $T$, and $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\,\|f\|, \|f\| = \|x\|\}, \qquad x \in E.$$

Let $g : E \to E$ be a single-valued mapping. Then, $g$ is called strongly accretive if for each $x$, $y$ in $E$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle g(x) - g(y), j(x-y)\rangle \geq \sigma\|x-y\|^2,$$

for some real constant $\sigma > 0$.

DEFINITION 2.1. *(See [11].) Let $A : D(A) \subset E \to 2^E$ be a set-valued mapping.*

(1) *The mapping $A$ is said to be accretive if, for any $x, y \in D(A)$, $u \in A(x)$, $v \in A(y)$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle u - v, j(x-y)\rangle \geq 0.$$

(2) *The mapping $A$ is said to be $m$-accretive if $A$ is accretive and $(I + \rho A)(D(A)) = E$ for every $\rho > 0$, where $I$ is the identity mapping.*

For a given $m$-accretive mapping $A : E \to 2^E$, $T, V : E \to CB(E)$ be two set-valued mappings, and a nonlinear mapping $N(\cdot,\cdot) : E \times E \to E$, we consider the problem of finding $u \in E$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w,y) + A(g(u)). \tag{2.1}$$

Problem (2.1) is called the generalized set-valued variational inclusion in Banach spaces.

Now, we consider some special cases of problem (2.1).

(1) If $T$,$V$ are the single-valued mappings, problem (2.1) can be replaced to finding $u \in H$ such that

$$0 \in N(T(u), V(u)) + A(g(u)). \tag{2.2}$$

(2) If $E = H$ is a Hilbert space and $A : D(A) = H \to H$ is a maximal monotone mapping, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 = N(w,y) + A(g(u)). \tag{2.3}$$

This problem is called the generalized set-valued variational inclusion, which was introduced and studied in [1].

(3) If $g \equiv I$, the identity mapping, then problem (2.1) is equivalent to finding $u \in E$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w,y) + A(u), \tag{2.4}$$

which is called the set-valued variational inclusion in Banach spaces.

(4) If $E = H$ is a Hilbert space and $A = \partial\phi$, the subdifferential of a proper convex lower semicontinuous function $\phi : H \to R \cup \{+\infty\}$, then problem (2.3) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w,y), v - g(u)\rangle \geq \phi(g(u)) - \phi(v), \tag{2.5}$$

for all $v \in H$. This problem is called the generalized set-valued mixed variational inequality, which was introduced and studied by Noor *et al.* [9].

Summing up the above arguments, it shows that, for a suitable choice of the mappings $T$, $V$, $A$, $g$, $N$, $\phi$, and the space $E$, we can obtain a number of known and new classes of variational inequalities, variational inclusions, and the corresponding optimization problems from the generalized set-valued variational inclusion in Banach spaces (2.1).

If $A : D(A) \subset E \to 2^E$ is an $m$-accretive mapping, then for a constant $\rho > 0$, the resolvent operator associated with $A$ is defined by

$$R_A(u) = (I + \rho A)^{-1}(u), \qquad \text{for all } u \in D(A),$$

where $I$ is the identity operator. It is well known that $R_A$ is a single-valued and nonexpansive mapping (see [11]).

REMARK 2.1. It is well known that if $E = E^* = H$ is a Hilbert space, then the notion of an accretive mapping coincides with that of a monotone mapping [11].

REMARK 2.2. It is also well known that if the single-valued operator $A : H \to H$ is maximal strongly monotone with constant $\alpha > 0$, then the resolvent operator $R_A = (I+\rho A)^{-1}$ is Lipschitz continuous with constant $1/(1 + \alpha\rho)$ where $\rho > 0$ is a constant [12].

DEFINITION 2.2. *Let $N(\cdot,\cdot) : E \times E \to E$ be a nonlinear mapping. For all $u_1, u_2 \in E$, the operator $N(\cdot,\cdot)$ is said to be Lipschitz continuous with respect to the first argument if there exists a constant $\beta > 0$ such that*

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \le \beta\|u_1 - u_2\|.$$

*In a similar way, we can define the Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument.*

DEFINITION 2.3. *Let $T : E \to CB(E)$ be a set-valued mapping and $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$. $T$ is said to be $\xi$-Lipschitzian continuous if, for any $x, y \in E$,*

$$H(Tx, Ty) \le \xi\|x - y\|,$$

*where $\xi > 0$ is a constant.*

In relation to problem (2.1), we consider the problem of finding $z, u \in E$, $w \in Tu$, $y \in Vu$ such that

$$N(w, y) + \rho^{-1}F_A z = 0. \tag{2.6}$$

Here $\rho > 0$ is a constant and $F_A = I - R_A$. Equations of type (2.6) are called the resolvent equations in Banach spaces.

The subdifferential of a function $f$ on $E$ is a map $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \ge f(x) + \langle y - x, x^*\rangle, \text{ for all } y \in E\}.$$

It is well known that $J(x)$ is the subdifferential of the function $(1/2)\|x\|^2$. An immediate consequence of this is the following lemma.

LEMMA 2.1. *Let $E$ be a real Banach space. Then, there exists $j(x + y) \in J(x + y)$ such that*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

*for all $x, y \in E$.*

If we use the technique given in [1], we can prove the following lemmas immediately.

LEMMA 2.2. *$(u, w, y)$ is a solution of (2.1) if and only if $(u, w, y)$ satisfies the relation*

$$g(u) = R_A(g(u) - \rho N(w, y)). \tag{2.7}$$

LEMMA 2.3. *The variational inclusion (2.1) has a solution $u \in E$, $w \in Tu$, $y \in Vu$ if and only if the resolvent equations (2.6) have a solution $z, u \in H$, $w \in Tu$, $y \in Vu$, where*

$$g(u) = R_A z, \tag{2.8}$$

*and*

$$z = g(u) - \rho N(w, y). \tag{2.9}$$

We now invoke Lemmas 2.2 and 2.3 to suggest the following algorithms for solving the generalized set-valued variational inclusion in Banach spaces (2.1).

**Algorithm 2.1**

For given $z_0 \in E$, we take $u_0 \in E$ such that

$$g(u_0) = R_A(z_0).$$

Let $w_0 \in Tu_0$, $y_0 \in Vu_0$, and $z_1 = g(u_0) - \rho N(w_0, y_0)$. For $z_1$, we take $u_1$ such that $g(u_1) = R_A z_1$. Then, by [13], there exist $w_1 \in Tu_1$ and $y_1 \in Vu_1$ such that

$$\|w_1 - w_0\| \le (1+1)H(Tu_1, Tu_0),$$
$$\|y_1 - y_0\| \le (1+1)H(Vu_1, Vu_0),$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$. By induction, we can obtain the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ as

$$\begin{aligned}
&g(u_n) = R_A z_n,\\
&\quad w_n \in Tu_n : \|w_{n+1} - w_n\| \le \left(1 + \frac{1}{n+1}\right) H(Tu_{n+1}, Tu_n),\\
&\quad y_n \in Vu_n : \|y_{n+1} - y_n\| \le \left(1 + \frac{1}{n+1}\right) H(Vu_{n+1}, Vu_n),\\
&z_{n+1} = g(u_n) - \rho N(w_n, y_n).
\end{aligned}$$

The following algorithm is called the perturbed Ishikawa iterative process.

**Algorithm 2.2**

For single-valued operators $T, V, g : E \to E$, define $Q : E \to E$ by

$$Qu = u - g(u) + R_A[g(u) - \rho N(Tu, Vu)].$$

For given $u_0 \in E$, the Ishikawa iterative scheme with error [14] is defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Q v_n + e_n,$$
$$v_n = (1 - \beta_n)u_n + \beta_n Q u_n + f_n,$$

where $\{e_n\}$,$\{f_n\}$ are two summable sequences in $E$, and $\{\alpha_n\}$,$\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying

$$0 \le \beta_n \le \alpha_n < 1, \qquad \sum_{n=0}^{\infty} \alpha_n = +\infty, \qquad \sum_{n=0}^{\infty} \alpha_n^2 < +\infty.$$

LEMMA 2.4. *(See [15].) Suppose $E$ is an arbitrary real Banach space and $T : E \to E$ is a strongly accretive and Lipschitz continuous operator. For a fixed $f \in E$, define $S : E \to E$ by $Sx = f + x - Tx$ for each $x \in E$. Let $\{f_n\}$, $\{e_n\}$ be two summable sequences in $E$, and $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences satisfying*

(i) $0 \le \beta_n \le \alpha_n < 1$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < +\infty$.

*Then, for any $x_0 \in E$, the iteration sequence $\{x_n\}$ in $E$ defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + e_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n S x_n + f_n,$$

*converges strongly to the unique solution $x^*$ of the equation $Tx = f$.*

# 3. MAIN RESULTS

THEOREM 3.1. *Let the operator $N(\cdot,\cdot)$ be Lipschitzian continuous with constant $\beta > 0$ with respect to the first argument and Lipschitz continuous with constant $\eta > 0$ with respect to the second argument. Let the single-valued operator $g : E \to E$ be Lipschitz continuous with constant $\delta > 0$ and $(g - I)$ be strongly accretive with constant $\sigma > 0$. Assume that $V : E \to CB(E)$ is $H$-Lipschitz continuous with constant $\xi > 0$, and $T : E \to CB(E)$ is $H$-Lipschitz continuous with constant $\mu > 0$. If $A : E \to 2^E$ is $m$-accretive and the following condition is satisfied:*

$$0 < \rho < \frac{\sqrt{1+2\sigma} - \delta}{\eta\xi + \beta\mu}, \tag{3.1}$$

*then there exist $u, z \in E$, $w \in Tu$, $y \in Vu$ satisfying the resolvent equation (2.6), and the iterative sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ generated by Algorithm 2.1 converge strongly to $z$, $u$, $w$, and $y$ in $E$, respectively.*

PROOF. From Algorithm 2.1, we have

$$\begin{aligned}\|z_{n+1} - z_n\| \le \|g(u_n) - g(u_{n-1})\| &+ \rho\|N(w_n, y_n) - N(w_n, y_{n-1})\| \\ &+ \rho\|N(w_n, y_{n-1}) - N(w_{n-1}, y_{n-1})\|.\end{aligned} \tag{3.2}$$

From the assumptions of $g$, $T$, and $V$, we get

$$\|g(u_n) - g(u_{n-1})\| \le \delta\|u_n - u_{n-1}\|, \tag{3.3}$$

$$\begin{aligned}\|N(w_n, y_n) - N(w_n, y_{n-1})\| &\le \eta\|y_n - y_{n-1}\| \le \eta\left(1 + \frac{1}{n}\right) H(Vu_n, Vu_{n-1}) \\ &\le \eta\left(1 + \frac{1}{n}\right)\xi\|u_n - u_{n-1}\|,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\|N(w_n, y_{n-1}) - N(w_{n-1}, y_{n-1})\| &\le \beta\|w_n - w_{n-1}\| \le \beta\left(1 + \frac{1}{n}\right) H(Tu_n, Tu_{n-1}) \\ &\le \beta\left(1 + \frac{1}{n}\right)\mu\|u_n - u_{n-1}\|.\end{aligned} \tag{3.5}$$

Also, using the strong accretivity of the operator $(g - I)$, (2.8), and Lemma 2.1, we find that

$$\begin{aligned}\|u_n - u_{n-1}\|^2 &= \|R_A z_n - R_A z_{n-1} - [g(u_n) - u_n - (g(u_{n-1}) - u_{n-1})]\|^2 \\ &\le \|R_A z_n - R_A z_{n-1}\|^2 - 2\langle g(u_n) - u_n - (g(u_{n-1}) - u_{n-1}), j(u_n - u_{n-1})\rangle \\ &\le \|z_n - z_{n-1}\|^2 - 2\sigma\|u_n - u_{n-1}\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}(1 + 2\sigma)\|u_n - u_{n-1}\|^2 &\le \|z_n - z_{n-1}\|^2, \\ \|u_n - u_{n-1}\|^2 &\le \frac{1}{1+2\sigma}\|z_n - z_{n-1}\|^2.\end{aligned} \tag{3.6}$$

Combining (3.2)–(3.6), we obtain

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \left[\delta + \rho\left(1 + \frac{1}{n}\right)(\eta\xi + \beta\mu)\right]\|u_n - u_{n-1}\| \\ &= \left[\delta + \rho\left(1 + \frac{1}{n}\right)(\eta\xi + \beta\mu)\right]\frac{1}{\sqrt{1+2\sigma}}\|z_n - z_{n-1}\| \\ &= \theta_n\|z_n - z_{n-1}\|,\end{aligned} \tag{3.7}$$

where

$$\theta_n = \left[\delta + \rho\left(1 + \frac{1}{n}\right)(\eta\xi + \beta\mu)\right]\frac{1}{\sqrt{1+2\sigma}}.$$

Let

$$\theta = [\delta + \rho(\eta\xi + \beta\mu)] \frac{1}{\sqrt{1+2\sigma}}.$$

Then, $\theta_n \to \theta$ as $n \to \infty$. It follows from (3.1) that $\theta \in (0,1)$. Hence, $0 \leq \theta_n \leq 1$ for $n$ sufficiently large. Therefore, (3.7) implies that $\{z_n\}$ is a Cauchy sequence in $E$. Since $E$ is a Banach space, there exists $z \in E$ such that $z_n \to z$ as $n \to \infty$. From (3.6), we know that the sequence $\{u_n\}$ is a Cauchy sequence in $E$, that is, there exists $u \in E$ with $u_n \to u$ as $n \to \infty$.

Now, we prove that $w_n \to w \in T(u)$ and $y_n \to y \in V(u)$.

In fact, it follows from Algorithm 2.1 that

$$\|w_n - w_{n-1}\| \leq \left(1 + \frac{1}{n}\right) H(Tu_n, Tu_{n-1}) \leq \left(1 + \frac{1}{n}\right) \mu \|u_n - u_{n-1}\|,$$
$$\|y_n - y_{n-1}\| \leq \left(1 + \frac{1}{n}\right) H(Vu_n, Vu_{n-1}) \leq \left(1 + \frac{1}{n}\right) \xi \|u_n - u_{n-1}\|,$$

that is, $\{w_n\}$ and $\{y_n\}$ are also Cauchy sequences in $E$. Let $w_n \to w \in E$, $y_n \to y \in E$. By using the continuity of the operators $T$, $N$, $g$, $V$, $R_A$, and Algorithm 2.1, we have

$$z_{n+1} = g(u_n) - \rho N(w_n, y_n) \to z = g(u) - \rho N(w,y), \qquad n \to \infty, \tag{3.8}$$
$$g(u_n) = R_A(z_n) \to g(u) = R_A(z), \qquad n \to \infty. \tag{3.9}$$

By (3.8), (3.9), Lemmas 2.2 and 2.3, we have

$$N(w,y) + \rho^{-1} F_A z = 0.$$

Finally, we prove that $w \in T(u)$ and $y \in V(u)$.

In fact,

$$\begin{aligned} d(w, T(u)) &= \inf\{\|w - z\| : z \in T(u)\} \leq \|w - w_n\| + d(w_n, T(u)) \\ &\leq \|w - w_n\| + H(T(u_n), T(u)) \leq \|w - w_n\| + \mu\|u_n - u\| \to 0, \qquad \text{as } n \to \infty. \end{aligned}$$

Hence, $w \in T(u)$. Similarly, $y \in V(u)$. This completes the proof of Theorem 3.1. ∎

REMARK 3.1. Theorem 3.1 is an improvement and extension of Theorem 3.2 in [1] from Hilbert spaces to real Banach spaces.

THEOREM 3.2. *Let the operator $N(\cdot,\cdot)$ be Lipschitzian continuous with constant $\beta > 0$ with respect to the first argument and Lipschitz continuous with constant $\eta > 0$ with respect to the second argument. Let the single-valued operator $g : E \to E$ be strongly accretive with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. Assume that $V : E \to E$ is single-valued Lipschitz continuous with constant $\xi > 0$, and $T : E \to E$ is single-valued Lipschitz continuous with constant $\mu > 0$. If $A : E \to 2^E$ is m-accretive and the following condition is satisfied:*

$$0 < \rho < \frac{\sigma - \delta}{\eta\xi + \beta\mu}, \qquad 0 < \sigma < 1, \tag{3.10}$$

*then the sequence $\{u_n\}$ generated by Algorithm 2.2 converges to the unique solution of problem (2.2).*

PROOF. It follows from Lemma 2.2 that problem (2.2) is equivalent to the resolvent equation

$$g(u) = R_A[g(u) - \rho N(Tu, Vu)], \tag{3.11}$$

that is,

$$u = u - g(u) + R_A[g(u) - \rho N(Tu, Vu)] = u - Q_1(u) = Q(u),$$

where $Q_1(u) = g(u) - R_A[g(u) - \rho N(Tu, Vu)]$. According to Lemma 2.4, we need only to prove $Q_1 : E \to E$ is strongly accretive with constant $\theta \in (0,1)$ and Lipschitz continuous. From the assumptions of the theorem, it is easy to see that the operator $Q_1$ is Lipschitz continuous. Let $u, v \in E$. Then, by the strong accretivity and Lipschitz continuity of $g$ and by the Lipschitz continuity of $N(\cdot,\cdot)$, we have

$$\begin{aligned}
\langle Q_1(u) - Q_1(v), j(u-v)\rangle &= \langle g(u) - g(v), j(u-v)\rangle \\
&\quad - \langle R_A[g(u) - \rho N(Tu, Vu)] - R_A[g(v) - \rho N(Tv, Vv)], j(u-v)\rangle \\
&\geq \sigma\|u-v\|^2 - \|R_A[g(u) - \rho N(Tu, Vu)] \\
&\quad - R_A[g(v) - \rho N(Tv, Vv)]\| \, \|j(u-v)\| \\
&\geq \sigma\|u-v\|^2 - (\|g(u) - g(v)\| + \rho\|N(Tu, Vu) - N(Tv, Vv)\|)\|u-v\| \\
&\geq \sigma\|u-v\|^2 - \delta\|u-v\|^2 - \rho\|N(Tu, Vu) - N(Tv, Vv)\| \, \|u-v\| \\
&\geq [\sigma - \delta - \rho(\beta\mu + \eta\xi)]\|u-v\|^2 = \theta\|u-v\|^2,
\end{aligned}$$

where $\theta = \sigma - \delta - \rho(\beta\mu + \eta\xi)$. Therefore, $Q_1 : H \to H$ is a strongly accretive operator with constant $\theta$ and from (3.10) it follows that $\theta \in (0,1)$. Thus, Lemma 2.4 is applicable where $f = 0$ and the proof is complete. ∎

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